Chapter 9: Integration

Exercise 9K

(1) let $I = \int (x+1)(x+3)^5 \, dx$

let $u = x+3$, $\therefore du = dx$

and $u - 2 = x+1$

so $I = \int (u-2) \cdot u^5 \, du = \int u^6 - 2u^5 \, du$

$= \frac{1}{7}u^7 - \frac{2}{6}u^6 + c$

$= \frac{1}{7}(x+3)^7 - \frac{1}{3}(x+3)^6 + c$

$= \frac{1}{21}(x+3)^6(3x+2) + c$

(2) let $I = \int \frac{1}{4+x^2} \, dx$

let $x = 2\tan\theta$, $\therefore dx = 2\sec^2\theta \, d\theta$

So $I = \int \frac{1}{4+4\tan^2\theta} \cdot 2\sec^2\theta \, d\theta$

$= \int \frac{2\sec^2\theta}{4\sec^2\theta} \, d\theta = \frac{1}{2}\int 1 \, d\theta$

$= \frac{1}{2}\theta + c$

$= \frac{1}{2}\tan^{-1}\frac{x}{2} + c$

③ let $I = \int \frac{x}{\sqrt{3-x}}\, dx$

let $u^2 = 3 - x \Rightarrow 2u\, du = -dx$ by implicit Differentiation

$\therefore -2u\, du = dx$

& $x = 3 - u^2$

$\therefore \quad I = \int \frac{3-u^2}{u} \cdot (-2u)\, du$

$= -\int 6 - 2u^2\, du$

$= \int 2u^2 - 6\, du$

$= \frac{2}{3}u^3 - 6u + c$

$= \frac{2}{3}(3-x)^{3/2} - 6(3-x)^{1/2} + c$

$= (3-x)^{1/2}\left[\frac{2}{3}(3-x) - 6\right]$

$= (3-x)^{1/2}\left[2 - \frac{2}{3}x - 6\right]$

$= -\frac{2}{3}(x+6)(3-x)^{1/2}$

(4) $I = \int x\sqrt{x+1}\,dx$

let $x+1 = u^2$ so $dx = 2u\,du$ by implicit differentiation

& $x = u^2 - 1$

so $I = \int (u^2-1) \cdot u \cdot 2u\,du$

$= \int 2u^4 - 2u^2\,du$

$= \frac{2}{5}u^5 - \frac{2}{3}u^3 + c$

$= \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + c$

$= (x+1)^{3/2}\left[\frac{2}{5}(x+1) - \frac{2}{3}\right]$

$= (x+1)^{3/2}\left[\frac{2}{5}x + \frac{2}{5} - \frac{2}{3}\right]$

$= (x+1)^{3/2}\left[\frac{2}{5}x - \frac{4}{15}\right]$

$= \frac{2}{15}(x+1)^{3/2}(3x-2)$

(5) let $I = \int \frac{2x+1}{(x-3)^6}\,dx$

let $u = x-3$, $\therefore du = dx$ & $x = u+3$

so $I = \int \frac{2(u+3)+1}{u^6}\,du$

$$\therefore \quad I = \int \frac{2u+7}{u^6}\, du$$

$$= \int \frac{2}{u^5} + \frac{7}{u^6}\, du$$

$$= -\frac{2}{4} u^{-4} - \frac{7}{5} u^{-5} + c$$

$$= -\frac{1}{2} \frac{1}{(x-3)^4} - \frac{7}{5} \frac{1}{(x-3)^5} + c$$

$$= \frac{1}{(x-3)^5} \left[-\frac{1}{2}(x-3) - \frac{7}{5} \right] + c$$

$$= \frac{1}{(x-3)^5} \left[-\frac{1}{2}x + \frac{3}{2} - \frac{7}{5} \right] + c$$

$$= \frac{1}{(x-3)^5} \left[-\frac{1}{2}x + \frac{1}{10} \right] + c = \frac{1-5x}{10(x-3)^5} + c$$

6 let $I = \int \frac{1}{\sqrt{1+x^2}}\, dx$

let $x = \tan\theta$, $\therefore$ $dx = \sec^2\theta\, d\theta$

So $I = \int \frac{1}{\sqrt{1+\tan^2\theta}} \cdot \sec^2\theta\, d\theta$

$$= \int \frac{1}{\sec\theta} \cdot \sec^2\theta\, d\theta = \int \sec\theta\, d\theta$$

Multiply by $\frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}$ to get an integral of the form $\frac{f'(x)}{f(x)}$:

So $I = \int \frac{\sec\theta\,(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta}\,d\theta$

$= \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta}\,d\theta$

let $z = \sec\theta + \tan\theta$ & continue, to get

$I = \ln|\sec\theta + \tan\theta| + c$

But $\sec\theta = \sqrt{1+\tan^2\theta}$ by trig id

$= \sqrt{1+x^2}$

So $I = \ln\left|\sqrt{1+x^2} + x\right| + c$

(7) let $I = \int 2x \cdot \sqrt{3x-4}\,dx$

let $u^2 = 3x-4$, $\therefore$ $2u\,du = 3\,dx$ by implicit differentiation

& $\frac{u^2+4}{3} = x$

So $I = \int \frac{2}{3}(u^2+4) \cdot u \cdot (\frac{2}{3}u)\,du$

$= \frac{4}{9}\int u^4 + 4u^2\,du$

$$\text{So } I = \frac{4}{9}\left(\frac{u^5}{5} + \frac{4}{3}u^3\right) + c$$

$$= \frac{4}{9}\left(\frac{1}{5}(3x-4)^{5/2} + \frac{4}{3}(3x-4)^{3/2}\right) + c$$

$$= \frac{4}{9}(3x-4)^{3/2}\left[\frac{1}{5}(3x-4) + \frac{4}{3}\right] + c$$

$$= \frac{4}{9}(3x-4)^{3/2}\left[\frac{3}{5}x - \frac{4}{5} + \frac{4}{3}\right] + c$$

$$= \frac{4}{9}(3x-4)^{3/2}\left[\frac{3}{5}x + \frac{8}{15}\right] + c$$

$$= \frac{4}{135}(3x-4)^{3/2}(9x+8) + c$$

(8) let $I = \int \frac{3}{25+4x^2}\,dx$

let $2x = 5\tan\theta$, $\therefore 2\,dx = 5\sec^2\theta\,d\theta$

$dx = \frac{5}{2}\sec^2\theta\,d\theta$

$$\therefore I = \int \frac{3}{25+25\tan^2\theta}\cdot\frac{5}{2}\sec^2\theta\,d\theta$$

$$= \frac{15}{2}\int \frac{1}{25\sec^2\theta}\cdot\sec^2\theta\,d\theta$$

$$= \frac{15}{50}\int 1\,d\theta = \frac{15}{50}\theta + c$$

$$= \frac{15}{50}\tan^{-1}\frac{2x}{5} + c$$

$$= \frac{3}{10}\tan^{-1}\left(\frac{2x}{5}\right) + c$$

(9) let $I = \int \frac{1}{\sqrt{(x^2+4x+3)}} dx$

$$= \int \frac{1}{\sqrt{(x+2)^2-1}} dx$$

let $x+2 = \sec\theta$, $\therefore$ $dx = \sec\theta . \tan\theta \, d\theta$

So $I = \int \frac{1}{\sqrt{\sec^2\theta - 1}} . \sec\theta . \tan\theta \, d\theta$

By $1+\tan^2\theta = \sec^2\theta$ we have $\sec^2\theta - 1 = \tan^2\theta$

So $I = \int \frac{1}{\tan\theta} . \sec\theta . \tan\theta \, d\theta$

$$= \int \sec\theta \, d\theta = \int \frac{\sec\theta(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta} d\theta$$

$$= \int \frac{\sec^2\theta + \sec\theta . \tan\theta}{\sec\theta + \tan\theta} d\theta$$

let $u = \sec\theta + \tan\theta$ & continue, to get

$$I = \ln|\sec\theta + \tan\theta| + C$$

$\sec\theta = x+2$ & $\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{(x+2)^2-1}$

So $I = \ln\left|(x+2) + \sqrt{(x+2)^2-1}\right| + C$

(10) Let $I = \int 2x(1-x)^7\,dx$

let $u = 1-x$, $\therefore du = -dx \Rightarrow -du = dx$

& $1-u = x$

So $I = \int 2(1-u)(u)^7(-1)\,du$

$= \int 2u^8 - 2u^7\,du$

$= \frac{2}{9}u^9 - \frac{2}{8}u^8 + c$

$= \frac{2}{9}(1-x)^9 - \frac{1}{4}(1-x)^8 + c$

(11) Let $I = \int \frac{1}{\sqrt{9-x^2}}\,dx$

let $x = 3\sin\theta$, $\therefore dx = 3\cos\theta\,d\theta$

So $I = \int \frac{1}{\sqrt{9-9\sin^2\theta}} \cdot 3\cos\theta\,d\theta$

$= \int \frac{1}{3\cos\theta} \cdot 3\cos\theta\,d\theta$

$= \int 1\,d\theta = \theta + c$

$= \sin^{-1}\frac{x}{3} + c$

(12) let $I = \int \frac{x+3}{(4-x)^5} \, dx$

let $u = 4-x$, $\therefore \; du = -dx \Rightarrow -du = dx$

So $I = \int -\frac{(4-u)+3}{u^5} \, du = \int -\frac{7-u}{u^5} \, du$

$= \int \frac{1}{u^4} - \frac{7}{u^5} \, du$

$= -\frac{1}{3} u^{-3} + \frac{7}{4} u^{-4} + c$

$= \frac{-1}{3(4-x)^3} + \frac{7}{4} \frac{1}{(4-x)^4} + c$

$= \frac{1}{(4-x)^4} \left[-\frac{1}{3}(4-x) + \frac{7}{4} \right] + c$

$= \frac{1}{(4-x)^4} \left[-\frac{4}{3} + \frac{x}{3} + \frac{7}{4} \right] + c$

$= \frac{1}{(4-x)^4} \cdot \left(\frac{x}{3} + \frac{5}{12} \right) + c$

$= \frac{5x+4}{12(4-x)^4} + c$

(13) use change of Variable / Substitution

let $I = \int e^{2x+3}\, dx$

let $u = 2x+3$, $\therefore$ $du = 2\,dx \Rightarrow \frac{1}{2}du = dx$

$$\therefore \quad I = \int \frac{1}{2} e^u \, du = \frac{1}{2} e^u + c$$

$$= \frac{1}{2} e^{2x+3} + c$$

(14) change of variable / Substitution

let $I = \int x\sqrt{2x^2-5}\, dx$

let $u = 2x^2-5$, $\therefore$ $du = 4x\, dx$

$\Rightarrow \frac{1}{4}du = x\, dx$

$$\text{So } I = \int \frac{1}{4} u^{\frac{1}{2}}\, du = \frac{2}{3}\cdot\frac{1}{4} u^{\frac{3}{2}} + c$$

$$= \frac{1}{6}(2x^2-5)^{\frac{3}{2}} + c$$

Could also have done

$x = \frac{\sqrt{5}}{\sqrt{2}}\sec\theta$, $\therefore$ $dx = \frac{\sqrt{5}}{\sqrt{2}}\sec\theta\cdot\tan\theta\, d\theta$

$$\text{So } I = \int \frac{\sqrt{5}}{\sqrt{2}}\cdot\sec\theta\sqrt{2\left(\frac{\sqrt{5}}{\sqrt{2}}\right)^2\sec^2\theta - 5}\cdot\frac{\sqrt{5}}{\sqrt{2}}\sec\theta\tan\theta\, d\theta$$

$$= \frac{5}{2}\int \sec\theta\sqrt{5\sec^2\theta - 5}\cdot\sec\theta\cdot\tan\theta\, d\theta$$

So $\quad I = \frac{5}{2} \int \sec\theta . \sqrt{5} . \tan\theta . \sec\theta . \tan\theta \, d\theta$

$$= \frac{5\sqrt{5}}{2} \int \sec^2\theta \tan^2\theta \, d\theta$$

Let $u = \tan\theta$ & continue, to get

$$I = \frac{5\sqrt{5}}{2} . \frac{1}{3} \tan^3\theta + c$$

Now $\quad x = \frac{\sqrt{5}}{\sqrt{2}} \sec\theta \quad$ so $\quad \frac{\sqrt{2}}{\sqrt{5}} x = \sec\theta$

So $\quad \frac{2}{5} x^2 - 1 = \sec^2\theta - 1 = \tan^2\theta$

So $\quad \tan\theta = \sqrt{\frac{2}{5}x^2 - 1}$

$$\therefore \quad I = \frac{5\sqrt{5}}{6} \left(\frac{2}{5} x^2 - 1\right)^{3/2} + c$$

This answer looks different to the previous answers but both are equivalent.
(Play around with the fraction inside the bracket so as to factorise $5\sqrt{5}$)

(15) use double angle formula

Let $I = \int \sin^2 3x \, dx$

From $\cos 2x = 1 - 2\sin^2 x$ we have $\cos 6x = 1 - 2\sin^2 3x$

$$\therefore \quad I = \int \frac{1}{2}(1 - \cos 6x) \, dx$$

$\therefore \quad I = \frac{1}{2}\left(x - \frac{1}{6}\sin 6x\right) + c$

$= \frac{1}{12}\left(6x - \sin 6x\right) + c$

(16) let $I = \int x e^{-x^2}\,dx$; use Substitution / change of Variables

So let $u = -x^2$, $\therefore \; du = -2x\,dx$

$\Rightarrow -\frac{1}{2}\,du = dx$

$\therefore \quad I = -\frac{1}{2}\int e^{u}\,du = -\frac{1}{2}e^{u} + c$

$= -\frac{1}{2}e^{-x^2} + c$

(17) use factor formula

let $I = \int \sin 3\theta \cdot \cos\theta\,d\theta = \frac{1}{2}\int 2\sin 3\theta\cos\theta\,d\theta$

Now, $\sin A + \sin B = 2\sin\frac{A+B}{2}\cdot\cos\frac{A-B}{2}$

So $\quad 3\theta = \frac{A+B}{2}$ & $\theta = \frac{A-B}{2}$

so $\quad A = 4\theta, \; B = 2\theta$

So $\quad I = \frac{1}{2}\int \sin 4\theta + \sin 2\theta\,d\theta$

$= -\frac{1}{8}\cos 4\theta - \frac{1}{4}\cos 2\theta + c$

(18) let $I = \int u(u+7)^9 \, du$; use change of variable / substitution

So let $z = u+7 \implies dz = du$

& $z - 7 = u$

$$\therefore I = \int (z-7) z^9 \, dz$$

$$= \int z^{10} - 7z^9 \, dz$$

$$= \frac{1}{11} z^{11} - \frac{7}{10} z^{10} + c$$

$$= \frac{1}{11}(u+7)^{11} - \frac{7}{10}(u+7)^{10} + c$$

$$= (u+7)^{10} \left[\frac{1}{11}(u+7) - \frac{7}{10} \right] + c$$

$$= (u+7)^{10} \left[\frac{1}{11} u - \frac{7}{110} \right] + c$$

$$= \frac{1}{110} (u+7)^{10} (10u - 7) + c$$

(19) use substitution / change of variables

let $I = \int \frac{x^2}{(x^3+9)^5} \, dx$

let $u = x^3 + 9$, $\therefore du = 3x^2 \, dx \Rightarrow \frac{1}{3} du = x^2 \, dx$

So $I = \frac{1}{3} \int \frac{1}{u^5} \, du = \frac{1}{3} \cdot \left(-\frac{1}{4}\right) \frac{1}{u^4} + c$

$$= -\frac{1}{12} \frac{1}{(x^3+9)^4} + c$$

(20) use Substitution / Change of Variables

let $I = \int \frac{\sin 2y}{1-\cos 2y}\, dy$

let $u = 1-\cos 2y$, $\therefore du = 2\sin 2y\, dy$

$\Rightarrow \frac{1}{2} du = \sin 2y\, dy$

$\therefore I = \int \frac{1}{2} \cdot \frac{1}{u}\, du = \frac{1}{2} \ln|u| + c$

$= \frac{1}{2} \ln|1-\cos 2y| + c$

(21) Integrate at sight

$I = \int \frac{1}{2x+7}\, dx = \frac{1}{2} \ln|2x+7| + c$

(to see how, use $u = 2x+7$ & continue)

(22) use a trig substitution

let $I = \int \frac{1}{\sqrt{1-u^2}}\, du$

Now let $u = \sin\theta$, $\therefore du = \cos\theta\, d\theta$

So $I = \int \frac{1}{\sqrt{1-\sin^2\theta}} \cdot \cos\theta\, d\theta$

$= \int \frac{1}{\cos\theta} \cdot \cos\theta\, d\theta = \int 1\, d\theta = \theta + c$

$= \sin^{-1} u + c$

(23) use change of variables / Substitution

let $I = \int \sin 3x . \sqrt{1+\cos 3x}\ dx$

now let $u = 1 + \cos 3x$, $\therefore$ $du = -3 \sin 3x\ dx$

so $-\frac{1}{3} du = \sin 3x\ dx$

$\therefore$ $I = -\frac{1}{3} \int \sqrt{u}\ du$

$= -\frac{1}{3} \cdot \frac{2}{3} u^{3/2} + c$

$= -\frac{2}{9} (1+\cos 3x)^{3/2} + c$

(24) use "By-Parts"

let $I = \int x . \sin 4x\ dx$

let $u = x$, $\therefore$ $\frac{du}{dx} = 1$

$\frac{dv}{dx} = \sin 4x$, $\therefore$ $v = -\frac{1}{4} \cos 4x$

so $I = -\frac{1}{4} x . \cos 4x - \left[\int -\frac{1}{4} \cos 4x\ dx \right]$

$= -\frac{1}{4} x . \cos 4x + \frac{1}{4} \int \cos 4x\ dx$

$= -\frac{1}{4} x . \cos 4x + \frac{1}{4} \cdot \frac{1}{4} \sin 4x + c$

$= -\frac{1}{4} x . \cos 4x + \frac{1}{16} \sin 4x + c$

(25) Complete the square on the denominator then use "$x+2 = \ldots$" as a substitution.

$$\text{let } I = \int \frac{x+2}{x^2+4x-5}\,dx$$

$$= \int \frac{x+2}{(x+2)^2-9}\,dx$$

let $x+2 = 3\sec\theta$, $\therefore$ $dx = 3\sec\theta.\tan\theta\,d\theta$

$$\text{So } I = \int \frac{3\sec\theta}{9\sec^2\theta-9}\cdot 3\sec\theta.\tan\theta\,d\theta$$

$$= \int \frac{3\sec\theta}{9\tan^2\theta}\cdot 3\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{\sec^2\theta}{\tan\theta}\,d\theta$$

$$= \ln|\tan\theta| + c$$ (use $z = \tan\theta$ as substitution)

But $x+2 = 3\sec\theta$, so $(x+2)^2 = 9\sec^2\theta$

$$= 9 + 9\tan^2\theta$$

so $\dfrac{(x+2)^2-9}{9} = \tan^2\theta$

$$\therefore I = \ln\left|\sqrt{\frac{(x+2)^2-9}{9}}\right| + c$$

So $I = \ln\left(\sqrt{(x+2)^2-9}\right) - \ln\left|\frac{1}{\sqrt{3}}\right| + c$

$= \frac{1}{2}\ln|x^2+4x-5| + k$, where $k = c - \ln\left|\frac{1}{\sqrt{3}}\right|$

Simpler still would be to use u substitution:

let $u = x^2+4x-5$, $\therefore$ $du = (2x+4)\,dx$

$\Rightarrow \frac{1}{2}\,du = (x+2)\,dx$

$\therefore I = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + c$

$= \frac{1}{2}\ln|x^2+4x-5| + c$

(26) use partial fractions

let $I = \displaystyle\int \frac{x+1}{x^2+4x-5}\,dx$

let $\dfrac{x+1}{x^2+4x-5} = \dfrac{x+1}{(x+5)(x-1)} = \dfrac{A}{x+5} + \dfrac{B}{x-1}$

$\therefore x+1 = A(x-1) + B(x+5)$

if $x=1$: $2 = 6B \Rightarrow B = \frac{1}{3}$

$x=-5$: $-4 = -6A \Rightarrow A = \frac{2}{3}$

So $I = \displaystyle\int \frac{2/3}{x+5} + \frac{1/3}{x-1}\,dx$

$= \frac{2}{3}\ln|x+5| + \frac{1}{3}\ln|x-1| + c = \frac{1}{3}\ln|(x+5)^2(x-1)| + c$

(27) use Substitution / change of variables

let $I = \int \frac{x+2}{(x^2+4x-5)^3} \, dx$

let $u = x^2+4x-5$, $\therefore du = (2x+4) \, dx$

$\Rightarrow \frac{1}{2} du = (x+2) \, dx$

$\therefore I = \frac{1}{2} \int \frac{1}{u^3} \, du = \frac{1}{2}(-2) \cdot u^{-2} + c$

$= - \frac{1}{(x^2+4x-5)^2} + c$

(28) Use Substitution / change of Variable

let $I = \int 3y \sqrt{9-y^2} \, dy$

let $u = 9-y^2$, $\therefore du = -2y \, dy$

$\therefore -\frac{3}{2} du = 3y \, dy$

So $I = -\frac{3}{2} \int \sqrt{u} \, du = -\frac{3}{2} \int u^{1/2} \, du$

$= -u^{3/2} + c$

$= -(9-y^2)^{3/2} + c$

(29) Use "By-parts"

let $I = \int e^{2x} \cos 3x \, dx$

let $u_1 = e^{2x}$, $\therefore \frac{du_1}{dx} = 2e^{2x}$

& $\frac{dv_1}{dx} = \cos 3x$, $\therefore v_1 = \frac{1}{3} \sin 3x$

So $I = \frac{1}{3} e^{2x} \sin 3x - \frac{2}{3} \int e^{2x} \sin 3x \, dx$

let $u_2 = e^{2x}$, $\therefore \frac{du_2}{dx} = 2e^{2x}$

& $\frac{dv_2}{dx} = \sin 3x$, $\therefore v = -\frac{1}{3} \cos 3x$

So $I = \frac{1}{3} e^{2x} \sin 3x - \frac{2}{3} \left[-\frac{1}{3} e^{2x} \cos 3x + \frac{2}{3} \int e^{2x} \cos 3x \, dx \right]$

$= \frac{1}{3} e^{2x} \sin 3x + \frac{2}{9} e^{2x} \cos 3x - \frac{4}{9} I$

So $I \left(1 + \frac{4}{9}\right) = \frac{1}{3} e^{2x} \cdot \sin 3x + \frac{2}{9} e^{2x} \cos 3x$

$I = \frac{9}{13} \left(\frac{1}{3} e^{2x} \cdot \sin 3x + \frac{2}{9} e^{2x} \cos 3x \right) + C$

$= \frac{1}{13} e^{2x} \left(3 \sin 3x + 2 \cos 3x \right) + C$

(30) use "by-parts"

let $I = \int \ln 5x \, dx = \int 1 \cdot \ln 5x \, dx$

Now let $u = \ln 5x$, $\therefore \frac{du}{dx} = \frac{5}{5x} = \frac{1}{x}$

& $\frac{dv}{dx} = 1$, $\therefore v = x$

So $I = x \cdot \ln|5x| - \int \frac{x}{x} \, dx$

$= x \cdot \ln|5x| - x + c$

$= x(\ln|5x| - 1) + c$

(31) use trig Identities Then Substitution / change of variable

let $I = \int \cos^3 2x \, dx$

$= \int \cos 2x \cdot \cos^2 2x \, dx$

$= \int \cos 2x (1 - \sin^2 2x) \, dx$

$= \int \cos 2x - \cos 2x \cdot \sin^2 2x \, dx$

$= \frac{1}{2} \sin 2x - \int \cos 2x \cdot \sin^2 2x \, dx$

let $u = \sin 2x$ & continue, to get

$I = \frac{1}{2} \sin 2x - \frac{1}{3} \sin^3 2x + c$

(32) change of Variable / Substitution

let $I = \int \operatorname{cosec}^2 x . e^{\cot x} \, dx$

let $u = \cot x$, $\therefore$ $du = -\operatorname{cosec}^2 x \, dx$

$\Rightarrow -du = \operatorname{cosec}^2 x \, dx$

So $I = \int -e^u \, du$

$= -e^u + c$

$= -e^{\cot x} + c$

(33) use change of variable / Substitution

let $I = \int \frac{\sin y}{\sqrt{7 + \cos y}} \, dy$

let $u = 7 + \cos y$, $\therefore$ $du = -\sin y \, dy$

$\Rightarrow -du = \sin y \, dy$

So $I = \int -\frac{1}{\sqrt{u}} \, du = \int -u^{-1/2} \, du$

$= -2 . u^{1/2} + c$

$= -2 (7 + \cos y)^{1/2} + c$

(34) use "by-parts"

let $I = \int x^2 \cdot e^x \, dx$

let $u_1 = x^2$, $\therefore \frac{du_1}{dx} = 2x$

& $\frac{dv_1}{dx} = e^x$, $\therefore v_1 = e^x$

So $I = x^2 . e^x - 2 \int x . e^x \, dx$

Now let $u_2 = x$, $\frac{du_2}{dx} = 1$

& $\frac{dv_2}{dx} = e^x$, $v_2 = e^x$

So $I = x^2 e^x - 2 \left[x e^x - \int e^x \, dx \right]$

$= x^2 e^x - 2x e^x + 2 e^x + c$

(35) use partial fractions

let $I = \int \frac{x}{x^2 - 4} \, dx = \int \frac{x}{(x-2)(x+2)} \, dx$

let $\frac{x}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$

So $x = A(x+2) + B(x-2)$

if $x = 2$: $2 = 4A \Rightarrow A = \frac{1}{2}$

$x = -2$: $-2 = -4B \Rightarrow B = \frac{1}{2}$

$$\therefore \quad I = \int \frac{1/2}{x-2} + \frac{1/2}{x+2}\, dx$$

$$= \tfrac{1}{2} \ln|x-2| + \tfrac{1}{2}\ln|x+2| + c$$

$$= \ln\left|\sqrt{x^2+4}\right| + c$$

(36) Simplify, then do Partial Fractions

let $I = \int \frac{x^2}{x^2-4}\, dx$

Now by long division

$$\begin{array}{r|l} & 1 \\ x^2-4 & x^2 \\ & x^2-4 \\ \hline & 4 \end{array}$$

So $I = \int 1 + \frac{4}{x^2-4}\, dx$

Now let $\frac{4}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}$

So $4 = A(x+2) + B(x-2)$

if $x=2$: $4 = 4A \Rightarrow A = 1$

$x=-2$: $4 = -4B \Rightarrow B = -1$

So $I = \int 1 + \frac{1}{x-2} - \frac{1}{x+2}\, dx$

$$= x + \ln|x-2| - \ln|x+2| + c$$

$$= x + \ln\left|\frac{x-2}{x+2}\right| + c$$

(37) Do Partial fractions

let $I = \int \frac{1}{x^2 - 4} \, dx = \int \frac{1}{(x-2)(x+2)} \, dx$

So let $\frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$

$\therefore \quad 1 = A(x+2) + B(x-2)$

let $x = 2: \quad 1 = 4A \Rightarrow A = \frac{1}{4}$

$x = -2: \quad 1 = -4B \Rightarrow B = -1/4$

So $I = \int \frac{1/4}{x-2} - \frac{1/4}{x+2} \, dx$

$= \frac{1}{4} \ln|x-2| - \frac{1}{4} \ln|x+2| + c$

$= \ln \left| \sqrt[4]{\frac{x-2}{x+2}} \right| + c$

(38) use factor formula of trig identities

let $I = \int \cos 4x \cdot \cos x \, dx = \frac{1}{2} \int 2 \cos 4x \cdot \cos x \, dx$

Now, $\cos A + \cos B = 2 \cos \frac{A+B}{2} \cdot \cos \frac{A-B}{2}$

So $\left. \begin{array}{l} 4x = \frac{A+B}{2} \quad \& \quad x = \frac{A-B}{2} \end{array} \right\} \Rightarrow A = 5x, \; B = 3x$

$$\therefore \quad I = \frac{1}{2}\int \cos 5x + \cos 3x \, dx$$

$$= \frac{1}{2}\cdot\frac{1}{5}\sin 5x + \frac{1}{2}\cdot\frac{1}{3}\sin 3x + c$$

$$= \frac{1}{30}\left(3\sin 5x + 5\sin 3x\right) + c$$

(39) use trig Identities Then Substitution / change of Variable

$$\text{Let } I = \int \sin^5 2\theta \, d\theta$$

$$= \int \sin 2\theta \cdot (\sin^2 2\theta)^2 \, d\theta$$

$$= \int \sin 2\theta \cdot (1 - \cos^2 2\theta)^2 \, d\theta$$

$$= \int \sin 2\theta \cdot (1 - 2\cos^2 2\theta + \cos^4 2\theta)\, d\theta$$

$$= \int \sin 2\theta - 2\sin 2\theta \cos^2 2\theta + \sin 2\theta \cos^4 2\theta \, d\theta$$

For 2^{nd} & 3^{rd} terms use $u = \cos 2\theta$ as sub & continue, to get

$$I = -\frac{1}{2}\cos 2\theta + \frac{1}{3}\cos^3 2\theta - \frac{1}{2}\frac{1}{5}\cos^5 2\theta \, d\theta$$

$$= \frac{1}{30}\cos 2\theta \left(+15 - 10\cos^2 2\theta + 3\cos^4 2\theta\right) + c$$

(40) trig Identity Then change of variable / Substitution

let $I = \int \cos^2 u \cdot \sin^3 u \, du$

$= \int \cos^2 u \cdot \sin^2 u \cdot \sin u \, du$

$= \int \cos^2 u \, (1 - \cos^2 u) \cdot \sin u \, du$

$= \int \cos^2 u \cdot \sin u - \cos^4 u \cdot \sin u \, du$

let $z = \cos u$ & continue, to get

$I = -\frac{1}{3} \cos^3 u + \frac{1}{5} \cos^5 u + c$

$= \frac{1}{15} \left(3 \cos^5 u - 5 \cos^3 u \right) + c$

(41) use trig identities & Substitution / change of variable

let $I = \int \tan^4 \theta \, d\theta$

$= \int \tan^2 \theta \cdot \tan^2 \theta \, d\theta$

$= \int \tan^2 \theta \, (\sec^2 \theta - 1) \, d\theta$

$= \int \tan^2 \theta \cdot \sec^2 \theta - \tan^2 \theta \, d\theta$

$= \int \tan^2 \theta \cdot \sec^2 \theta - \sec^2 \theta + 1 \, d\theta$

use $u = \tan \theta$ for the 1st term & continue, to get

$$I = \frac{1}{3}\tan^3\theta - \tan\theta + \theta + c$$

(42) use trig identities & change of variable / substitution

let $I = \int \tan^5\theta \, d\theta$

$$= \int \tan\theta \left(\tan^2\theta\right)^2 d\theta$$

$$= \int \tan\theta \left(\sec^2\theta - 1\right)^2 d\theta$$

$$= \int \tan\theta . \left(\sec^4\theta - 2\sec^2\theta + 1\right) d\theta$$

$$= \int \tan\theta \cdot \sec^4\theta - 2\tan\theta\sec^2\theta + \tan\theta \, d\theta \quad (*)$$

$$= \int \frac{\sin\theta}{\cos^5\theta} - 2\tan\theta\sec^2\theta + \frac{\sin\theta}{\cos\theta} \, d\theta$$

let $u_1 = \cos\theta$ for 1st & 3rd terms, & let $u_2 = \tan\theta$ for 2nd term & continue to get

$$I = \frac{1}{4}\cos^{-4}\theta - \tan^2\theta - \ln|\cos\theta| + c$$

OR $(*)$ can be written as

$$I = \int \tan\theta \sec^2\theta . \sec^2\theta - 2\tan\theta \cdot \sec^2\theta + \tan\theta \, d\theta$$

$$= \int \tan\theta \left(1 + \tan^2\theta\right) \cdot \sec^2\theta - 2\tan\theta . \sec^2\theta + \frac{\sin\theta}{\cos\theta} \, d\theta$$

$$= \int \tan\theta . \sec^2\theta + \tan^3\theta\sec^2\theta - 2\tan\theta . \sec^2\theta + \frac{\sin\theta}{\cos\theta} \, d\theta$$

$$I = \int \tan^3\theta \cdot \sec^2\theta - \tan\theta \cdot \sec^2\theta + \frac{\sin\theta}{\cos\theta}\, d\theta$$

$$= \frac{1}{4}\tan^4\theta - \frac{1}{2}\tan^2\theta - \ln|\cos\theta| + c$$

Although the two answers look different they are Equivalent, & both can be differentiated & simplified to give $\tan^5\theta$.

(43) trig Substitution & Normal "u" Substitution

$$\text{let } I = \int \frac{1-2x}{\sqrt{1-x^2}}\, dx = \int \frac{1}{\sqrt{1-x^2}} - \frac{2x}{\sqrt{1-x^2}}\, dx$$

For the 1st term use a trig substitution, for the 2nd term use a "u" substitution (That is why we split I into two integrals)

So, let $x = \sin\theta$, $\therefore\ dx = \cos\theta\, d\theta$

& let $u = 1 - x^2$, $\therefore\ du = -2x\, dx$

$$\text{So } I = \int \frac{\cos\theta}{\sqrt{1-\sin^2\theta}}\, d\theta + \int \frac{1}{\sqrt{u}}\, du$$

$$= \int \frac{\cos\theta}{\cos\theta}\, d\theta + \int u^{-1/2}\, du$$

$$= \int 1\, d\theta + \int u^{-1/2}\, du$$

$$= \theta + 2u^{1/2} + c = \sin^{-1}x + 2(1-x^2)^{1/2} + c$$

(44) change of variable / Substitution

let $I = \int \frac{1}{u \ln u} \, du$

let $z = \ln u$, $\therefore$ $dz = \frac{1}{u} \, du$

So $I = \int \frac{1}{z} \, dz = \ln|z| + c$

$= \ln \left| \ln|u| \right| + c$

(45) By-parts

let $I = \int y^2 \cdot \cos 3y \, dy$

let $u_1 = y^2$, $\therefore$ $\frac{du_1}{dy} = 2y$

& $\frac{dv_1}{dy} = \cos 3y$, $\therefore$ $v_1 = \frac{1}{3} \sin 3y$

So $I = \frac{1}{3} y^2 \cdot \sin 3y - \frac{2}{3} \int y \cdot \sin 3y \, dy$

Now let $u_2 = y$, $\therefore$ $\frac{du_2}{dy} = 1$

& $\frac{dv_2}{dy} = \sin 3y$, $\therefore$ $v_2 = -\frac{1}{3} \cos 3y$

So $I = \frac{1}{3} y^2 \cdot \sin 3y - \frac{2}{3} \left[-\frac{1}{3} y \cdot \cos 3y + \int \frac{1}{3} \cos 3y \, dy \right]$

$= \frac{1}{3} y^2 \cdot \sin 3y + \frac{2}{9} y \cdot \cos 3y - \frac{2}{9} \cdot \frac{1}{3} \cdot \sin 3y + c$

$= \frac{1}{27} \left(9y^2 \cdot \sin 3y + 6y \cdot \cos 3y - 2 \cdot \sin 3y \right) + c$

(46) Substitution / Change of Variables

let $I = \int \frac{\sec^2 x}{1-\tan x}\, dx$

let $u = (1-\tan x)\, dx$, $\therefore\ du = -\sec^2 x\, dx$

$\Rightarrow -du = \sec^2 x\, dx$

So $I = -\int \frac{1}{u}\, du = -\ln|u| + c$

$= -\ln|1-\tan x| + c$

(47) Substitution / Change of Variable

let $I = \int x.\sqrt{7+x^2}\, dx$

let $u = 7+x^2$, $\therefore\ du = 2x\, dx$

$\Rightarrow \frac{1}{2} du = x\, dx$

So $I = \frac{1}{2}\int \sqrt{u}\, du = \frac{1}{2}.\frac{2}{3} u^{3/2} + c$

$= \frac{1}{3}(7+x^2)^{3/2} + c$

(OR) let $x = \sqrt{7}\tan\theta$, $\therefore\ dx = \sqrt{7}\sec^2\theta\, d\theta$

& $I = \int \sqrt{7}.\tan\theta.\sqrt{7+7\tan^2\theta}\,.\sqrt{7}\sec^2\theta\, d\theta$

$= 7\int \tan\theta \sqrt{7\sec^2\theta}\,.\sec^2\theta\, d\theta$

$= 7\sqrt{7}\int \tan\theta.\sec^3\theta\, d\theta$

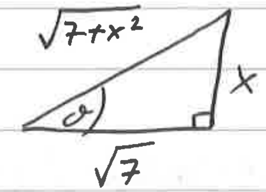
$$\therefore \quad I = 7\sqrt{7} \int \frac{\sin\theta}{\cos^4\theta}\, d\theta$$

$$= \frac{7\sqrt{7}}{3} \frac{1}{\cos^3\theta} + c \qquad \text{(let } u = \cos\theta \text{ \& continue as usual)}$$

$$= \frac{7\sqrt{7}}{3} \cdot \sec^3\theta + c$$

This answer is the same as the previous Ans, & can be proved as follows:

$x = \sqrt{7}\tan\theta \;,\; \therefore \; \frac{x}{\sqrt{7}} = \tan\theta$:



So by diagram $\cos\theta = \frac{\sqrt{7}}{\sqrt{7+x^2}}$

$$\therefore \quad \sec\theta = \frac{\sqrt{7+x^2}}{\sqrt{7}} = \left(\frac{7+x^2}{7}\right)^{1/2}$$

$$\text{So} \quad \sec^3\theta = \left(\frac{7+x^2}{7}\right)^{3/2}$$

$$\therefore \quad I = \frac{7\sqrt{7}}{3} \cdot \left(\frac{7+x^2}{7}\right)^{3/2} + c$$

Factolise out the denominator "7"

$$\therefore \quad I = \frac{7\sqrt{7}}{3} \cdot \frac{1}{(\sqrt{7})^3} (7+x^2)^{3/2} + c$$

$$= \frac{1}{3} (7+x^2)^{3/2} + c .$$

(48) use Substitution / change of variable

let $I = \int \sin(5\theta - \frac{\pi}{4})\, d\theta$

let $u = 5\theta - \frac{\pi}{4}$, $\therefore du = 5\, d\theta$

$\Rightarrow \frac{1}{5} du = d\theta$

So $I = \frac{1}{5} \int \sin u \, du = -\frac{1}{5} \cos u + c$

$= -\frac{1}{5} \cos(5\theta - \frac{\pi}{4}) + c$

(49) use Substitution / change of variable

let $I = \int \cos\theta . \ln(\sin\theta)\, d\theta$ (assume $\sin\theta > 0$)

let $u = \sin\theta$, $\therefore du = \cos\theta\, d\theta$

So $I = \int \ln u \, du = \int 1 . \ln u \, du$

let $u_1 = \ln u$, $\therefore \frac{du_1}{du} = \frac{1}{u}$

& $\frac{dv_1}{du} = 1$, $\therefore V_1 = u$

So $I = u \ln u - \int \frac{1}{u} . u \, du$

$= u . \ln u - u + c$

$= \sin\theta . \ln|\sin\theta| - \sin\theta + c$

$= \sin\theta \{ \ln|\sin\theta| - 1 \} + c$

(50) use Substitution / Change of Variables

let $I = \int \sec^2 u . e^{\tan u} \, du$

let $z = \tan u$, $\therefore \; dz = \sec^2 u \, du$

So $I = \int e^z . dz = e^z + c$

$= e^{\tan u} + c$

(51) Substitution / Change of Variable

let $I = \int \frac{x}{(3-x)^7} \, dx$

let $u = 3-x$, $\therefore \; du = -dx$

$\Rightarrow -du = dx$

$\therefore \; x = 3-u$

So $I = -\int (3-u) . \frac{1}{u^7} \, dx$

$= -\int \frac{3}{u^7} - \frac{1}{u^6} \, du$

$= \frac{3}{6} \frac{1}{u^6} - \frac{1}{5} \frac{1}{u^5} + c$

$= \frac{1}{2} \frac{1}{(3-x)^6} - \frac{1}{5} \frac{1}{(3-x)^5} + c$

$= \frac{1}{(3-x)^6} \left\{ \frac{1}{2} - \frac{1}{5}(3-x) \right\} + c$

$$\therefore \quad I = \frac{1}{(3-x)^6}\left(\frac{1}{2} - \frac{3}{5} - \frac{x}{5}\right)$$

$$= \frac{1}{(3-x)^6}\left(-\frac{1}{10} + \frac{x}{5}\right)$$

$$= \frac{1}{10(3-x)^6}\cdot(2x-1)$$

(52) use Substitution / Change of Variable

let $I = \int \tan^2 x \cdot \sec^2 x \, dx$

let $u = \tan x$, $\therefore \quad du = \sec^2 x \, dx$

So $I = \int u^2 \, du = \frac{1}{3}u^3 + c$

$$= \frac{1}{3}\tan^3 x + c.$$